Journal of Applied Mechanics and Technical Physics, No. 4, 1994

DISSIPATIVE TWO-VELOCITY HYDRODYNAMICS

S. M. Shugrin

UDC 532.5

In [1], new equations were obtained for "ideal" two-velocity hydrodynamics. In this article, we use Onsager's general principle to construct a dissipative system.

1. "Ideal" Two-Velocity Hydrodynamics. Let the state of the system at the point (t, x^1, x^2, x^3) be characterized by the set $(\rho_{(1)}, v_{(1)}, \rho_{(2)}, v_{(2)}, s)$, where $\rho_{(j)} \geq 0$ is the density of the j-th component, $\rho_{(1)} + \rho_{(2)} > 0$; $v_{(j)} = (v^k{}_{(j)})$ is the velocity of the j-th component; s is entropy. We put

$$\rho \equiv \rho_{(1)} + \rho_{(2)},\ \varkappa \equiv \rho_{(1)}/\rho,\ v \equiv \varkappa v_{(1)} + (1-\varkappa)v_{(2)},\ w \equiv v_{(1)} - v_{(2)}.$$

Now we can assume that the state of the system is characterized by the set $u \equiv (\rho, \varkappa, s, v, w)$. In accordance with [1], the minimal two-velocity system has the form

$$L^0(u) \equiv \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k}(\rho v^k) = 0; \tag{1.1a}$$

$$L^j(u) \equiv \frac{\partial}{\partial t}(\rho v^j) + \frac{\partial}{\partial x^k}[\rho v^j v^k + \varkappa(1-\varkappa)\rho w^j w^k + p\delta^{jk}] = 0; \tag{1.1b}$$

$$L_E(u) \equiv \frac{\partial}{\partial t}[\rho(\varepsilon + v^2/2)] + \frac{\partial}{\partial x^k}\{\rho v^k[\varepsilon + v^2/2 + p/\rho] + \rho w^k[\varkappa(1-\varkappa)(v\cdot w) + \varkappa(1-\varkappa)\mu_{(1)0} + T\zeta_{(12)}/\rho \tag{1.2}$$

$$+ \varkappa(1-\varkappa)(1-2\varkappa)w^2/2]\} = 0;$$

$$l^0_{(1)}(u) \equiv \frac{\partial}{\partial t}(\varkappa\rho) + \frac{\partial}{\partial x^k}[\varkappa\rho v^k + \varkappa(1-\varkappa)\rho w^k] - q_{(1)} = 0; \tag{1.3a}$$

$$l^j_{(1)} \equiv \frac{\partial}{\partial t}[\varkappa\rho v^j + \varkappa(1-\varkappa)\rho w^j] + \frac{\partial}{\partial x^k}[\varkappa\rho v^j v^k + \varkappa(1-\varkappa)\rho(v^j w^k + v^k w^j) + \varkappa(1-\varkappa)^2\rho w^j w^k] \tag{1.3b}$$

$$+ \varkappa\frac{\partial p}{\partial x^j} + \varkappa(1-\varkappa)\rho\frac{\partial \mu_{(1)0}}{\partial x^j} + \zeta_{(12)}\frac{\partial T}{\partial x^j} - q_{(1)}v^j + fw^j = 0.$$

Equations (1.1) represent the law of conservation of mass-momentum for the system as a whole, Eqs. (1.2) give the law of energy conservation, and Eqs. (1.3) are the law of mass-momentum balance for component 1. Here, $\varepsilon(\rho, \varkappa, s, w^2)$ is the internal energy of the two-velocity system determined — in accordance with hypothesis I in [1] — by means of the expressions

$$\rho(\varepsilon + v^2/2) = \rho\varepsilon_0(\rho, \varkappa, s) + \rho_{(1)}v^2_{(1)}/2 + \rho_{(2)}v^2_{(2)}/2,$$
$$d\varepsilon_0 = Tds - pd(1/\rho) + \mu_{(1)0}d\varkappa,$$

i.e. $\varepsilon = \varepsilon_0 + \varkappa(1 - \varkappa w^2/2$. From this it follows

$$d\varepsilon = Tds - pd(1/\rho) + [\mu_{(1)0} + (1-2\varkappa)w^2/2]d\varkappa + \varkappa(1-\varkappa)w^k dw^k. \tag{1.4}$$

Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 4, pp. 59-68, July-August, 1994. Original article submitted July 16, 1993; revision submitted September 20, 1993.

0021-8944/94/3504-0538$12.50

The functions $\varepsilon_0(\rho, \varkappa, s)$, $\xi_{(12)}(\rho, \varkappa, s, w^2)$, $q_{(1)}(\rho, \varkappa, s, w^2)$, $f(\rho, \varkappa, s, w^2)$ are assigned. The law of entropy conservation follows from (1.1)-(1.4)

$$
\begin{gathered}
L_s(u) \equiv \frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x^k}(\rho s v^k + \zeta_{(12)} w^k) - \theta = 0, \\
\theta \equiv (f w^2 - \mu_{(1)} q)/T, \ \mu_{(1)} \equiv \mu_{(1)0} - (1 - 2\varkappa) w^2/2.
\end{gathered}
\tag{1.5}
$$

The other equations of "ideal" two-velocity hydrodynamics (1.1)-(1.3) can be written in the form

$$
l^{\rho}_{(1)}(u) \equiv \frac{\partial}{\partial t}(\rho_{(1)}) + \frac{\partial}{\partial x^k}(\rho_{(1)} v^k_{(1)}) - q_{(1)} = 0; \tag{1.6a}
$$

$$
\begin{gathered}
l^j_{(1)}(u) \equiv \frac{\partial}{\partial t}(\rho_{(1)} v^j_{(1)}) + \frac{\partial}{\partial x^k}(\rho_{(1)} v^j_{(1)} v^k_{(1)}) \\
+ \varkappa \frac{\partial p}{\partial x^j} + \varkappa(1 - \varkappa)\rho \frac{\partial \mu_{(1)0}}{\partial x^j} + \zeta_{(12)} \frac{\partial T}{\partial x^j} - q_{(1)} v^j + f(v^j_{(1)} - v^j_{(2)}) = 0;
\end{gathered}
\tag{1.6b}
$$

$$
l^{\rho}_{(2)}(u) \equiv \frac{\partial}{\partial t}(\rho_{(2)}) + \frac{\partial}{\partial x^k}(\rho_{(2)} v^k_{(2)}) - q_{(2)} = 0; \tag{1.7a}
$$

$$
\begin{gathered}
l^j_{(2)}(u) \equiv \frac{\partial}{\partial t}(\rho_{(2)} v^j_{(2)}) + \frac{\partial}{\partial x^k}(\rho_{(2)} v^j_{(2)} v^k_{(2)}) \\
+ (1 - \varkappa)\frac{\partial p}{\partial x^j} + \varkappa(1 - \varkappa)\rho \frac{\partial \mu_{(2)0}}{\partial x^j} + \zeta_{(21)} \frac{\partial T}{\partial x^j} - q_{(2)} v^j + f(v^j_{(2)} - v^j_{(1)}) = 0;
\end{gathered}
\tag{1.7b}
$$

$$
\begin{gathered}
\zeta_{(21)} \equiv - \zeta_{(12)}, \ \mu_{(2)0} \equiv - \mu_{(1)0}, \ q_{(2)} \equiv - q_{(1)}; \\
L_E(u) \equiv \frac{\partial}{\partial t}[\rho\varepsilon_0 + \rho_{(1)} v^2_{(1)}/2 + \rho_{(2)} v^2_{(2)}/2] \\
+ \frac{\partial}{\partial x^k}\{\rho_{(1)} v^k_{(1)}[\varepsilon_0 + v^2_{(1)}/2 + p/\rho + (1 - \varkappa)\mu_{(1)0} + T\zeta_{(12)}/\rho_{(1)}] \\
+ \rho_{(2)} v^k_{(2)}[\varepsilon_0 + v^2_{(2)}/2 + p/\rho + \varkappa\mu_{(2)0} + T\zeta_{(21)}/\rho_{(2)}]\} = 0.
\end{gathered}
\tag{1.8}
$$

Equations (1.6) represent the law of mass-momentum balance for component 1, Eqs. (1.7) give the same for component 2, and Eq. (1.8) is the energy conservation law. Adding (1.6) and (1.7), we obtain the mass-momentum conservation law for the system as a whole:

$$
L^0(u) \equiv \frac{\partial}{\partial t}(\rho_{(1)} + \rho_{(2)}) + \frac{\partial}{\partial x^k}(\rho_{(1)} v^k_{(1)} + \rho_{(2)} v^k_{(2)}) = 0; \tag{1.9a}
$$

$$
L^j(u) \equiv \frac{\partial}{\partial t}(\rho_{(1)} v^j_{(1)} + \rho_{(2)} v^j_{(2)}) + \frac{\partial}{\partial x^k}(\rho_{(1)} v^j_{(1)} v^k_{(1)} + \rho_{(2)} v^j_{(2)} v^k_{(2)} + p\delta^{jk}) = 0. \tag{1.9b}
$$

Equations (1.6), (1.8), and (1.9) are different forms of Eqs. (1.3), (1.2), and (1.1), respectively. With allowance for (1.4), we obtain (1.5) from (1.6)-(1.8).

If $w \approx 0$, then Eqs. (1.1), (1.2), and (1.3a) become the equations of one-velocity hydrodynamics:

$$
L^0(u) = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k}(\rho v^k) = 0; \tag{1.10a}
$$

$$
L^j(u) = \frac{\partial}{\partial t}(\rho v^j) + \frac{\partial}{\partial x^k}[\rho v^j v^k + p\delta^{jk}] = 0; \tag{1.10b}
$$

$$
L_E(u) = \frac{\partial}{\partial t}[\rho(\varepsilon_0 + v^2/2)] + \frac{\partial}{\partial x^k}[\rho v^k(\varepsilon_0 + v^2/2 + p/\rho)] = 0; \tag{1.11}
$$

$$
l^{\rho}_{(1)}(u) = \frac{\partial}{\partial t}(\varkappa\rho) + \frac{\partial}{\partial x^k}(\varkappa\rho v^k) - q_{(1)} = 0. \tag{1.12}
$$

If we multiply (1.10a) by $q_0 \equiv -(\gamma_0 - v^2)/T$, where $\gamma_0 = \varepsilon_0 + p/\rho - Ts - \varkappa\mu_{(1)0}$, and if we multiply (1.10b) by $q_j \equiv -v^j/T$, (1.11) by $1/T$, and (1.12) by $\mu_{(1)0}/T$, we obtain the entropy conservation law

$$
L_s(u) = \frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x^k}(\rho s v^k) = 0. \tag{1.13}
$$

2. Onsager's General Principle. Let there be a system of equations written in the form of conservation laws:

$$\frac{\partial\varphi_s^0(u)}{\partial t} + \frac{\partial\varphi_s^k(u)}{\partial x^k} - f_s(u) = 0 \ (s = 1, \dots ,m). \tag{2.1}$$

Here, $u \equiv (u_1,\dots,u_m)$; $\varphi_s^\alpha(u)$, $f_s(u) \in R$. We assume that the system is complete in the sense of the word used in [2]. This means that there are $q^s(u)$ such that multiplication of (2.1) by q^s yields yet another conservation law

$$\frac{\partial\Phi^0(u)}{\partial t} + \frac{\partial\Phi^k(u)}{\partial x^k} - F(u) = 0; \tag{2.2}$$

$$d\Phi^\alpha(u) = q^s d\varphi_s^\alpha, \ F = q^s f_s, \tag{2.3}$$

Meanwhile, the transformation $u \to q(u)$ ($q \equiv (q^1,\dots, q^m)$) is one-to-one, i.e. the inverse transformation $q \to u$ is unambiguously determined. The quantities (q^s) are called **integrating factors**, the conservation laws (2.1) **fundamental conservation laws**, and law (2.2) the **closing conservation law**. For example, with suitable enumeration, the integrating factors are as follows for system (1.10-1.12)

$$q_0 = -(\gamma_0 - v^2/2)/T, \ q_j = -v^j/T, \ q_4 = 1/T, \ q_5 = -\mu_{(1)0}/T. \tag{2.4}$$

Let system (2.1) be invariant relative to the complete Galilean group Γ [1] and, in particular, relative to the group SO(3), i.e. the group of rotations. We further assume that the initial set of integrating factors ($q^1,\dots q^m$) can be represented in the form of the set ($z_{<1>},\dots, z_{<r>}$), where each $z_{<p>}$ is transformed in accordance with the tensor rule in transformations from SO(3), i.e. each is a certain SO(3)-tensor (scalar, vector, ...). Thus, in accordance with (2.4), for (1.10-1.12) the set q includes three SO(3)-scalars: $-(\gamma_0 - v^2/2)/T$, $1/T$, and $-\mu_{(1)0}/T$, as well as one SO(3)-vector $-v/T$.

Below, we will be discussing a group of coordinate transformations. The group SO(3), supplemented by reflections, is transformed into the group O(3) of orthogonal transformations of the space R^3.

There are now two main possibilities. First, each $z_{<p>}$ is transformed as a tensor of the corresponding type in all transformations from O(3) (and will be referred to as a **Euclidean tensor of the corresponding type**). Second, certain $z_{<k>}$ may be transformed as relative tensors of weight 1 [3] (and will then be referred to as **relative Euclidean tensors of the corresponding type**). We recall that $T \equiv (T_{k_1\dots k_q}{}^{j_1\dots j_p})$ is called a relative tensor of weight 1 of type (p, q) if in the coordinate transformation $x \to \bar{x}(x)$ the quantity T is transformed according to the rule $T(x) \to \bar{T}(\bar{x})$:

$$\bar{T}^{j_1\dots j_p}_{k_1\dots k_q} = \det\left(\frac{\partial\bar{x}}{\partial x}\right)\frac{\partial\bar{x}^{j_1}}{\partial x^{a_1}}\cdots\frac{\partial\bar{x}^{j_p}}{\partial x^{a_p}}\frac{\partial x^{b_1}}{\partial\bar{x}^{k_1}}\cdots\frac{\partial x^{b_q}}{\partial\bar{x}^{k_q}}T_{b_1\dots b_q}{}^{a_1\dots a_p}$$

In particular, ζ is a relative Euclidean scalar if in a mirror transformation of the coordinates $\zeta \to -\zeta$.

System (2.1) will be called a system of the **Onsager type** in the first case and a system of the **Onsager—Kazimir type** in the second case. For example, system (1.10-1.12) is an Onsager system. For these systems, dissipative terms are introduced so that symmetry conditions are satisfied for them (see below). In the case of Onsager—Kazimir systems, conditions of antisymmetry are satisfied for the relations connecting the tensors and the relative tensors.

We will further assume that (2.1) is of the Onsager type. In changing over from (2.1) to a **dissipative Onsager system**, dissipative terms are introduced into (2.1) to yield the equations [4, 5]

$$\frac{\partial\varphi_s^0(u)}{\partial t} + \frac{\partial\varphi_s^k(u)}{\partial x^k} - f_s(u) = \frac{\partial}{\partial x^k}\left[L_{sr}^{kj}(u)\frac{\partial q^r}{\partial x^j}\right], \tag{2.5}$$

Meanwhile, the following **symmetry conditions** are satisfied

$$L_{sr}^{kj}(u) = L_{rs}^{jk}(u). \tag{2.6}$$

We multiply (2.5) by q^s and use (2.3). Instead of (2.2) we then have

$$\frac{\partial\Phi^0(u)}{\partial t} + \frac{\partial\Phi^k(u)}{\partial x^k} - F(u) = \sigma + \frac{\partial}{\partial x^k}\left[q^r L^{kj}_{sr}\frac{\partial q^r}{\partial x^j}\right]; \tag{2.7}$$

$$\sigma \equiv - L^{kj}_{sr}(u)\frac{\partial q^r}{\partial x^j}\frac{\partial q^s}{\partial x^k}. \tag{2.8}$$

The function σ must also satisfy the requirements of dissipation and invariance. Since the entropy conservation law ($\Phi^0 = \rho s$) is the closing law in the given case, we adopt the following conditions.

Dissipation Conditions:

$$\sigma \geq 0. \tag{2.9}$$

Invariance Condition: the function σ, determined by Eq. (2.8), is a Galilean scalar (see the tensor classification in [1]).

With allowance for (2.6), we write Eq. (2.5) in the form

$$\frac{\partial\varphi^0_s}{\partial t} + \frac{\partial\varphi^k_s}{\partial x^k} - f_s = \frac{\partial}{\partial x^k}\left[-\frac{1}{2}\frac{\partial\sigma}{\partial(\partial q^s/\partial x^k)}\right], \tag{2.10}$$

while $L^{kj}_{sr} = -\frac{1}{2}\frac{\partial^2\sigma}{\partial(\partial q^s/\partial x^k)\partial(\partial q^r/\partial x^j)}$.

Conversely, let us assign the function $\sigma(u, \partial q/\partial x)$. We assume that it is quadratically dependent on the derivatives. If we take

$$L^{kj}_{sr} \equiv -\frac{1}{2}\frac{\partial^2\sigma}{\partial(\partial q^s/\partial x^k)\partial(\partial q^r/\partial x^j)},$$

then

$$L^{kj}_{sr} = -\frac{1}{2}\frac{\partial^2\sigma}{\partial(\partial q^s/\partial x^k)\partial(\partial q^r/\partial x^j)} = -\frac{1}{2}\frac{\partial^2\sigma}{\partial(\partial q^r/\partial x^j)\partial(\partial q^s/\partial x^k)} = L^{jk}_{rs},$$

i.e. symmetry condition (2.6) is satisfied.

Thus, **with a closing entropy conservation law**, the overall procedure for introducing dissipative terms into Onsager system (2.1) reduces to the following:

1) find integrating factors ($q^1, \ldots, q^m$). If $\Phi^0(u)$ and $\varphi^0(u)$ are known, it is most convenient to find these factors on the basis of (2.3), i.e. from the relation

$$d\Phi^0 = q^s d\varphi^0_s; \tag{2.11}$$

2) establish the general form of Galilean invariant $\sigma(u, \partial q/\partial x)$, which is quadratically dependent on the derivatives ($\partial q^2/\partial x^k$);

3) write the conditions for satisfaction of dissipation condition (2.9);

4) use (2.10) to finally determine the dissipative Onsager system.

The above method of introducing dissipative terms is then extended to systems for which the transformation $q \to u$ (the inverse of $u \to q(u)$) is multiple-valued (i.e. systems for which there may be a finite set of inverse images u for certain q). Such systems may be encountered when phase transformations occur [1]. The same principle will be used below in two-velocity hydrodynamics, where not all of the equations are in the form of conservation laws.

Note 2.1. It is often necessary to satisfy the "Curie principle" — the principle of the "conservation of causal symmetry in the symmetry of the effects." Here, this principle is interpreted as follows: if q^r and q^s are introduced as components into tensors of different ranks, then $L^{jk}_{rs} = 0$ (the cases usually examined here are those in which L^{jk}_{rs} has diagonal symmetry relative to the indices jk, i.e. $L^{jk}_{rs} \equiv L_{rs}\delta^{jk}$). As far as we know, this condition was not considered by Curie and in fact constitutes a another principle — an "asymmetry principle." If L^{jk}_{rs} are constants or depend only on scalars, then the Curie principle follows from the requirement of the invariance of σ. If $L^{jk}_{rs}(u)$ is dependent on vectors and other tensors of rank

≥ 1, then, in the general case, the Curie principle is invalid as it is presently formulated (see Part 3 as well). Thus, an accurate general formulation of the Curie principle requires that σ be invariant.

Note 2.2. In a discussion of this subject, A. N. Konovalov noted that (2.5) is sometimes best written in the form

$$\frac{\partial \varphi^0(u)}{\partial t} + \frac{\partial \varphi^k(u)}{\partial x^k} + f(u) + (l^* S l)q = 0,$$
$$\varphi^\alpha = (\varphi_1^\alpha, \ldots, \varphi_m^\alpha),\ f = (f_1, \ldots, f_m),\ q = (q^1, \ldots, q^m), \tag{2.12}$$

where l is a linear differential operator; l^* is the adjoint operator; S(u) is a symmetric operator.

The operators l and l^* are expressed through a standard set of elementary tensor operators (differential forms) for symmetric and skew-symmetric tensors satisfying invariance conditions. They are found on the basis of considerations relating to tensor classification (see [1]). In this regard, form (2.12) seems promising for describing and analyzing the general form of dissipative Onsager systems.

Note 2.3. A clearer analysis of Onsager systems is obtained in the case of relativistic systems, where the requirement of Lorentz invariance is used regularly in place of the weaker SO(3)-invariance [5]. A similar approach is impossible for Galilean systems due to the absence of a nondegenerate metric in the 4-dimensional Galilean coordinate state.

Note 2.4. We are formulating only a general principle in this section. Additional limitations that reflect the specifics of the given problem or are introduced to simplify the latter may be present.

3. Dissipative Two-Velocity Systems. Let us turn to equations (1.1)-(1.3), (1.5), first calculating the integrating factors for them. To do this, we change (1.4) to the form

$$d(\rho s) = -\frac{(\gamma_0 - \varkappa^2 w^2/2)}{T} d\rho + \frac{1}{T} d(\rho\varepsilon)$$
$$-\frac{1}{T}(\mu_{(1)0} - (1 - 2\varkappa)w^2/2)d(\rho\varkappa) - \frac{w^k}{T} d[\varkappa(1 - \varkappa)\rho w^k], \tag{3.1}$$
$$\gamma_0 \equiv \varepsilon_0 + p/\rho - Ts - \varkappa\mu_{(1)0}.$$

Thus, in accordance with [1], the quantities ρ, $\rho\varepsilon$, $\rho\kappa$, $\kappa(1 - \kappa)\rho w$ are canonical (thermodynamic) variables of the first order, while $\rho s = \Phi(\rho, \rho\varepsilon, \rho\varkappa, \varkappa)\rho w$ is the first canonical (thermodynamic) potential. We now convert (3.1) to the complete differential form and designate $E \equiv \rho(\varepsilon + v^2/2)$. Here, ρs is a function of nine quantities: ρ, ρv, E, $\rho\varkappa$, $\varkappa\rho v + \varkappa(1 - \varkappa)\rho w$. It follows from (3.1) that

$$d(\rho s) = q_0 d\rho + q_j d(\rho v^j) + q_4 dE + q_5 d(\rho\varkappa) + q_{5+j} d[\varkappa\rho v^j + \varkappa(1 - \varkappa)\rho w^j]; \tag{3.2}$$

$$q_0 \equiv -\frac{1}{T}[\gamma_0 - \varkappa^2 w^2/2 + \varkappa(v \cdot w) - v^2/2] = -\frac{1}{T}[\gamma_0 - v_{(2)}^2/2]; \tag{3.3a}$$

$$q_j \equiv -\frac{1}{T}(v^j - \varkappa w^j) = -\frac{1}{T}v_{(2)}^j; \tag{3.2b}$$

$$q_4 \equiv 1/T; \tag{3.3c}$$

$$q_5 \equiv \frac{1}{T}[\mu_{(1)0} - (1 - 2\varkappa)w^2/2 - (v \cdot w)]$$
$$= -\frac{1}{T}[\mu_{(1)0} - v_{(1)}^2/2 + v_{(2)}^2/2]; \tag{3.3d}$$

$$q_{5+j} \equiv -\frac{1}{T}w^j. \tag{3.3e}$$

The integrating factors are determined (compare (1.1)-(1.3), (1.5), and (3.2)-(3.3) with (2.1)-(2.2), (2.11)). Now the problem involves determining the invariant σ. First we formulate general restrictions on σ in addition to those mentioned in Part 2. Dissipative terms are not usually introduced into mass conservation law (1.1a), so σ should not depend on $(\partial q_0/\partial x^k)$. We expand the tensor $(\partial v^i/\partial x^k)$ into the sum of the symmetric diagonal tensor $(\delta^{ik}\partial v^1/\partial x^1)$ — which has no trace and is made up of the components $(1/2)(\partial v^i/\partial x^k) + (\partial v^k/\partial x^i) - (2/3)(\delta^{ik}\partial v^i/\partial x^i)$ — and a skew-symmetric tensor with the components $\Omega_{ik} \equiv (1/2)(\partial v^i/\partial x^k) - \partial v^k/\partial x^i$. The invariant σ is independent of (Ω_{ik}), since no dissipation occurs when the fluid rotates as a rigid body [6]. However, in cases in which such motions are prohibited by a heterogeneous medium, are we to conclude that σ is still independent of (Ω_{ik})?! It should be noted that it does follow in any way from the foregoing that σ is independent of $\omega_{ik} \equiv (1/2)(\partial w^i/\partial x^k - \partial w^k/\partial x^i)$.

The general expression that we finally obtain for σ turns out to be very awkward. Thus, for the sake of simplification we take the expression

$$
\begin{gathered}
\sigma \equiv \frac{\chi}{T^2}\frac{\partial T}{\partial x^j}\frac{\partial T}{\partial x^j} + \frac{2\alpha}{T}\frac{\partial T}{\partial x^j}\frac{\partial \xi}{\partial x^j} + \lambda\frac{\partial \xi}{\partial x^j}\frac{\partial \xi}{\partial x^j} \\
+ \sum_{s,j,k}\left\{\frac{\eta_{(s)}}{2T}\left(\frac{\partial v^j_{(s)}}{\partial x^k} + \frac{\partial v^k_{(s)}}{\partial x^j} - \frac{2}{3}\delta^{jk}\frac{\partial v^l_{(s)}}{\partial x^l}\right)^2 + \frac{\zeta_{(s)}}{T}\left(\frac{\partial v^l_{(s)}}{\partial x^l}\right)^2\right\}, \\
\xi \equiv \mu_{(1)0}/T.
\end{gathered}
\tag{3.4}
$$

The quantities χ, α, λ, $\eta_{(s)}$, and $\zeta_{(s)}$ in (3.4) may depend only on the scalars ρ, $\varkappa$, s, and w^2 (also see superfluid dynamics in [6]).

We write the condition of non-negativity of (3.4) in the form

$$
\begin{gathered}
d(\rho s) = \sum_k \{h^0_{(k)} d\rho_{(k)} + h^j_{(k)} d(\rho_{(k)} v^j_{(k)})\} + \tau dE, \\
h^0_{(1)} \equiv -\frac{1}{T}[\varepsilon_0 + p/\rho - Ts + (1-\varkappa)\mu_{(1)0} - v^2_{(1)}/2], \; h^j_{(1)} \equiv -\frac{1}{T}v^j_{(1)}, \\
h^0_{(2)} \equiv -\frac{1}{T}[\varepsilon_0 + p/\rho - Ts - \varkappa\mu_{(1)0} - v^2_{(2)}/2], \; h^j_{(2)} \equiv -\frac{1}{T}v^j_{(2)}, \\
\tau \equiv 1/T.
\end{gathered}
$$

Now it is convenient to calculate the dissipative terms for system (1.6)-(1.8). We find the following integrating factors for it:

$$
\eta_{(s)} \geqslant 0, \; \zeta_{(s)} \geqslant 0, \; \chi \geqslant 0, \; \lambda \geqslant 0, \; \chi\lambda \geqslant \alpha^2.
$$

The derivatives in (3.4) should then be expressed in terms of derivatives of $h^\alpha{}_{(k)}$. For example, we have

$$
\xi = -h^0_{(1)} + v^2_{(1)}/2T + h^0_{(2)} - v^2_{(2)}/2T,
$$

from which

$$
\begin{gathered}
\frac{\partial \xi}{\partial x^k} = -\frac{\partial h^0_{(1)}}{\partial x^k} + v^j_{(1)}\frac{\partial}{\partial x^k}(v^j_{(1)}/T) + \frac{\partial h^0_{(2)}}{\partial x^k} - v^j_{(2)}\frac{\partial}{\partial x^k}(v^j_{(2)}/T) \\
- (v^2_{(1)}/2 - v^2_{(2)}/2)\partial\tau/\partial x^k.
\end{gathered}
$$

After calculating, we arrive at the equations

$$
l^0_{(1)}(u) = \frac{\partial}{\partial x^k}\left\{\lambda\frac{\partial \xi}{\partial x^k} + \frac{\alpha}{T}\frac{\partial T}{\partial x^k}\right\}; \tag{3.6a}
$$

$$
\begin{gathered}
l^j_{(1)}(u) = \frac{\partial}{\partial x^k}\left\{\lambda v^j_{(1)}\frac{\partial \xi}{\partial x^k} + \frac{\alpha}{T}v^j_{(1)}\frac{\partial T}{\partial x^k}\right. \\
\left. + \eta_{(1)}\left(\frac{\partial v^j_{(1)}}{\partial x^k} + \frac{\partial v^k_{(1)}}{\partial x^j} - \frac{2}{3}\delta^{jk}\frac{\partial v^l_{(1)}}{\partial x^l}\right) + \zeta_{(1)}\delta^{jk}\frac{\partial v^l_{(1)}}{\partial x^l}\right\};
\end{gathered}
\tag{3.6b}
$$

$$
l^0_{(2)}(u) = \frac{\partial}{\partial x^k}\left\{-\lambda\frac{\partial \xi}{\partial x^k} - \frac{\alpha}{T}\frac{\partial T}{\partial x^k}\right\}; \tag{3.7a}
$$

$$
l^j_{(2)}(\mu) = \frac{\partial}{\partial x^k}\left\{-\lambda v^j_{(2)}\frac{\partial \xi}{\partial x^k} - \frac{\alpha}{T}v^j_{(2)}\frac{\partial T}{\partial x^k} + \eta_{(2)}\left(\frac{\partial v^j_{(2)}}{\partial x^k} + \frac{\partial v^k_{(2)}}{\partial x^j} - \frac{2}{3}\delta^{jk}\frac{\partial v^l_{(2)}}{\partial x^l}\right) + \zeta_{(2)}\delta^{jk}\frac{\partial v^l_{(2)}}{\partial x^l}\right\}; \tag{3.7b}
$$

$$L_E(u) = \frac{\partial}{\partial x^k}\left\{\chi\frac{\partial T}{\partial x^k} + \alpha T\left[\frac{\partial \xi}{\partial x^k} + \frac{1}{T^2}(v^2_{(1)}/2 - v^2_{(2)}/2)\frac{\partial T}{\partial x^k}\right] + \lambda(v^2_{(1)}/2 - v^2_{(2)}/2)\frac{\partial \xi}{\partial x^k} + \sum_q v^j_{(q)}\left[\eta_{(q)}\left(\frac{\partial v^k_{(q)}}{\partial x^j} + \frac{\partial v^j_{(q)}}{\partial x^k} - \frac{2}{3}\delta^{jk}\frac{\partial v^l_{(q)}}{\partial x^l}\right) + \zeta_{(q)}\delta^{jk}\frac{\partial v^l_{(q)}}{\partial x^l}\right]\right\}. \tag{3.8}$$

We multiply (3.6a) by $h^0_{(1)}$, (3.6b) by $h^j_{(1)}$, (3.7a) by $h^0_{(2)}$, (3.7b) by $h^j_{(2)}$, and (3.8) by τ and add. As a result, we obtain

$$L_s(u) = \sigma + \frac{\partial}{\partial x^k}\left\{\frac{\chi}{T}\frac{\partial T}{\partial x^k} + \alpha T\frac{\partial}{\partial x^k}(\xi/T) - \lambda\xi\frac{\partial \xi}{\partial x^k}\right\}. \tag{3.9}$$

Equations (3.6)-(3.8), along with the equation (3.9) that follows from them, constitute the complete system of equations of dissipative two-velocity hydrodynamics. Adding (3.6) and (3.7), we have

$$L^0(u) = 0; \tag{3.10a}$$

$$L^j(u) = \frac{\partial}{\partial x^k}\left\{\sum_q\left[\eta_{(q)}\left(\frac{\partial v^j_{(q)}}{\partial x^k} + \frac{\partial v^k_{(q)}}{\partial x^j} - \frac{2}{3}\delta^{jk}\frac{\partial v^l_{(q)}}{\partial x^l}\right) + \zeta_{(q)}\delta^{jk}\frac{\partial v^l_{(q)}}{\partial x^l}\right]\right\}. \tag{3.10b}$$

If $w \approx 0$, then (3.10), (3.8), and (3.6a) become the equations of one-velocity hydrodynamics:

$$L^0(u) = 0; \tag{3.11a}$$

$$L^j(u) = \frac{\partial}{\partial x^k}\left\{\eta\left(\frac{\partial v^j}{\partial x^k} + \frac{\partial v^k}{\partial x^j} - \frac{2}{3}\delta^{jk}\frac{\partial v^l}{\partial x^l}\right) + \zeta\delta^{jk}\frac{\partial v^l}{\partial x^l}\right\},$$
$$\eta \equiv \eta_{(1)} + \eta_{(2)},\ \zeta \equiv \zeta_{(1)} + \zeta_{(2)}; \tag{3.11b}$$

$$L_E(u) = \frac{\partial}{\partial x^k}\left\{\chi\frac{\partial T}{\partial x^k} + \alpha T\frac{\partial \xi}{\partial x^k} + v^j\left[\eta\left(\frac{\partial v^j}{\partial x^k} + \frac{\partial v^k}{\partial x^j} - \frac{2}{3}\delta^{jk}\frac{\partial v^l}{\partial x^l}\right) + \zeta\delta^{jk}\frac{\partial v^l}{\partial x^l}\right]\right\}; \tag{3.12}$$

$$\ell_{(1)}(u) = \frac{\partial}{\partial x^k}\left\{\lambda\frac{\partial \xi}{\partial x^k} + \frac{\alpha}{T}\frac{\partial T}{\partial x^k}\right\}. \tag{3.13}$$

Note 3.1. Surprisingly, no attempt has apparently yet been made to prove that friction was actually introduced into Navier–Stokes equations (3.11b) and (3.12) in order to satisfy the Onsager principle. Such a proof would simultaneously show that the "Curie principle" is invalid here, the reason obviously being the fact that the corresponding kinetic coefficients L^{jk}_{sr} depend on the vector v (in canonical notation! Incidentally, for the energy equation, this is already clear from (3.12)).

4. Nontraditional Approach. The method used in Part 3 to obtain dissipative terms corresponds to the classical Onsager principle in the general form described in Part 2. There is now no doubt as to its correctness for one-velocity systems (under the condition, of course, that the dissipative terms are linear relative to the derivatives). However, since there is some doubt in the case of two-velocity systems, below we will describe an alternative method of representing the dissipative terms.

The momentum balance equations for the components (1.6b) and (1.7b) are nondivergent. The force caused by total pressure $\partial p/\partial x$ is distributed among the components in proportion to mass: $\varkappa$ and $(1 - \varkappa)$. This is probably also true for the other forces acting on the system (and entering into the conservation law for total momentum). Conversely, the equations for the system as a whole — conservation laws for mass, momentum, energy, and other quantities (if they exist) — will generally retain their divergent structure. Proceeding on the basis of these heuristic considerations, we write the following equations (compare with (3.6)-(3.8))

$$\ell_{(1)}(u) = \frac{\partial}{\partial x^k}\left\{\lambda\frac{\partial \xi}{\partial x^k} + \frac{\alpha}{T}\frac{\partial T}{\partial x^k}\right\}; \tag{4.1a}$$

$$\ell^j_{(1)}(u) = \frac{\partial}{\partial x^k}\left\{\lambda v^j_{(1)}\frac{\partial \xi}{\partial x^k} + \frac{\alpha}{T}v^j_{(1)}\frac{\partial T}{\partial x^k}\right\} \tag{4.1b}$$

$$+ \varkappa\frac{\partial}{\partial x^k}\left\{\eta\left(\frac{\partial v^j}{\partial x^k} + \frac{\partial v^k}{\partial x^j} - \frac{2}{3}\delta^{jk}\frac{\partial v^l}{\partial x^l}\right) + \zeta\delta^{jk}\frac{\partial v^l}{\partial x^l}\right\};$$

$$\ell^0_{(2)}(u) = \frac{\partial}{\partial x^k}\left\{-\lambda\frac{\partial \xi}{\partial x^k} - \frac{\alpha}{T}\frac{\partial T}{\partial x^k}\right\}; \tag{4.2a}$$

$$\ell^j_{(2)}(u) = \frac{\partial}{\partial x^k}\left\{-\lambda v^j_{(2)}\frac{\partial \xi}{\partial x^k} - \frac{\alpha}{T}v^j_{(2)}\frac{\partial T}{\partial x^k}\right\} \tag{4.2b}$$

$$+ (1-\varkappa)\frac{\partial}{\partial x^k}\left\{\eta\left(\frac{\partial v^j}{\partial x^k} + \frac{\partial v^k}{\partial x^j} - \frac{2}{3}\delta^{jk}\frac{\partial v^l}{\partial x^l}\right) + \zeta\delta^{jk}\frac{\partial v^l}{\partial x^l}\right\};$$

$$L_E(u) = \frac{\partial}{\partial x^k}\left\{\chi\frac{\partial T}{\partial x^k} + \alpha T\left[\frac{\partial \xi}{\partial x^k} + \frac{1}{T^2}(v^2_{(1)}/2 - v^2_{(2)}/2)\frac{\partial T}{\partial x^k}\right]\right. \tag{4.3}$$

$$+ \lambda(v^2_{(1)}/2 - v^2_{(2)}/2)\frac{\partial \xi}{\partial x^k}$$

$$\left. + v^j\left[\eta\left(\frac{\partial v^j}{\partial x^k} + \frac{\partial v^k}{\partial x^j} - \frac{2}{3}\delta^{jk}\frac{\partial v^l}{\partial x^l}\right) + \zeta\delta^{jk}\frac{\partial v^l}{\partial x^l}\right]\right\}.$$

Adding (4.1) and (4.2), we obtain the mass-momentum conservation law for the system as a whole:

$$L^0(u) = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k}(\rho v^k) = 0; \tag{4.4a}$$

$$L^j(u) = \frac{\partial}{\partial x^k}\left\{\eta\left(\frac{\partial v^j}{\partial x^k} + \frac{\partial v^k}{\partial x^j} - \frac{2}{3}\delta^{jk}\frac{\partial v^l}{\partial x^l}\right) + \zeta\delta^{jk}\frac{\partial v^l}{\partial x^l}\right\}. \tag{4.4b}$$

Here, η, ζ, χ, and α depend only on ρ, κ, s, and w^2 (or another equivalent set of scalars). We multiply (4.1a) by $h^0_{(1)}$, (4.1b) by $h^j_{(1)}$, (4.2a) by $h^0_{(2)}$, (4.2b) by $h^j_{(2)}$, and (4.3) by τ and add the products. This gives us

$$L_s(u) = \bar{\sigma} + \frac{\partial}{\partial x^k}\left\{\frac{\chi}{T}\frac{\partial T}{\partial x^k} + \alpha T\frac{\partial}{\partial x^k}(\xi/T) - \lambda\xi\frac{\partial \xi}{\partial x^k}\right\},$$

$$\bar{\sigma} \equiv \frac{\chi}{T^2}\frac{\partial T}{\partial x^j}\frac{\partial T}{\partial x^j} + \frac{2\alpha}{T}\frac{\partial T}{\partial x^j}\frac{\partial \xi}{\partial x^j} + \lambda\frac{\partial \xi}{\partial x^j}\frac{\partial \xi}{\partial x^j}$$

$$+ \sum_{j,k}\left\{\frac{\eta}{2T}\left(\frac{\partial v^j}{\partial x^k} + \frac{\partial v^k}{\partial x^j} - \frac{2}{3}\delta^{jk}\frac{\partial v^j}{\partial x^j}\right)^2 + \frac{\zeta}{T}\left(\frac{\partial v^j}{\partial x^j}\right)^2\right\}.$$

It is still not certain which of the two constructions of the two-velocity dissipative equations is correct (although variant (4.1-4.4) seems more natural). Additional arguments will be needed to make a final choice. Ultimately, it will be necessary to correctly generalize Onsager's principle to two-velocity systems. However, it is first necessary to recognize the problem, and illumination of the latter has been one of the main goals of the present investigation.

REFERENCES

1. S. M. Shugrin, "Two-velocity hydrodynamics and thermodynamics," Prikl. Mekh. Tekh. Fiz., No. 4 (1994).
2. S. M. Shugrin, "Conservation laws, invariance, and equations of gas dynamics," ibid., No. 2 (1989).
3. I. S. Sokol'nikov, Tensor Analysis [in Russian], Nauka, Moscow (1971).
4. S. K. Godunov, "Interesting class of quasilinear systems," Dokl. Akad. Nauk SSSR, No. 3 (1961).
5. S. M. Shugrin, "Equations of dissipative Lorentz hydrodynamics," Din. Sploshnoi Sredy, **96** (1990).
6. L. D. Landau and E. M. Lifshitz, Hydrodynamics [in Russian], Nauka, Moscow (1986).